HW 3 P43

(2a) $z^3+z+1 = x^3+3x^2yi+3x(yi)^2+(yi)^3+x+iy+1$

$= (x^3-3xy^2+x+1)+i(3x^2y-y^3+y).$

(2b) $\dfrac{\bar{z}^2}{z} = \dfrac{(x-iy)^2}{(x+iy)}$

$= \dfrac{(x-iy)^3}{x^2+y^2}$

$= \dfrac{x^3-3xy^2}{x^2+y^2} + i\,\dfrac{y^3-3x^2y}{x^2+y^2}$

~~3a~~ (3) $f = x^2-y^2-2y+i(2x-2xy)$

$= \left(\dfrac{z+\bar{z}}{2}\right)^2 - \left(\dfrac{z-\bar{z}}{2i}\right)^2 - 2\left(\dfrac{z-\bar{z}}{2i}\right) + i\left(z+\bar{z} - (z+\bar{z})\cdot\dfrac{(z-\bar{z})}{2i}\right)$

$= \dfrac{1}{4}\left(z^2+2z\bar{z}+\bar{z}^2\right) + \dfrac{1}{4}\left(z^2-2z\bar{z}+\bar{z}^2\right)$

$+i(z+\bar{z}) + \dfrac{i}{2}\left(2z+2\bar{z}+(z^2-\bar{z}^2)i\right)$

$= \bar{z}^2 + 2iz.$

(4) $f = z + \frac{1}{z}$

$= re^{i\theta} + \frac{1}{r} e^{-i\theta}$

$= \left(r + \frac{1}{r}\right) \cos\theta + i\left(r - \frac{1}{r}\right) \sin\theta$

P54-55

(1a) let $\varepsilon > 0$, if $|z - z_0| < \varepsilon$, then

$|Re(z) - R(z_0)| \leq |z - z_0| < \varepsilon$

(1b) let $\varepsilon > 0$, if $|z - z_0| < \varepsilon$, then

$|\bar{z} - \bar{z}_0| = |z - z_0| < \varepsilon$.

(1c) let $\varepsilon > 0$, if $|z - 0| < \varepsilon$, then

$\left|\frac{\bar{z}^2}{z}\right| = \frac{|z|^2}{|z|} = |z| < \varepsilon$

(5) if $z = x \in \mathbb{R}$ (real axis), $f(z) = \left(\frac{x}{x}\right)^2 = 1$

if $z = iy$, $y \in \mathbb{R}$ (Imaginary axis),

$f(z) = \left(\frac{iy}{-iy}\right)^2 = 1$

if $z = x + ix$ (the line $x = y$),

$f(z) = \left(\frac{x + ix}{x - ix}\right)^2 = \left(\frac{1+i}{1-i}\right)^2 = \left(\frac{-2i}{2}\right)^2 = -1$

The limit approaching 0 on real axis and the line $x = y$ ~~do~~ are not the same, thus the limit does not exist.

7 directly from the suggestion $\left(\big| |f(z)| - |w_0| \big| \le |f(z) - w_0| \right)$

11 a If $c = 0$, $\displaystyle \lim_{z \to 0} \frac{1}{\frac{\frac{a}{z} + b}{d}} = 0$.

11 b If $c \neq 0$. $\displaystyle \lim_{z \to \infty} T(z) = \lim_{z \to 0} T\left(\frac{1}{z}\right)$

$$= \lim_{z \to 0} \frac{a + bz}{c + dz}$$

$$= \frac{a}{c},$$

$$\lim_{z \to -\frac{d}{c}} \frac{1}{T(z)} = \lim_{z \to -\frac{d}{c}} \frac{c + dz}{a + bz} = 0.$$

P61 1 $\displaystyle \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\Delta z (2z + \Delta z)}{\Delta z}$$

$$= 2z.$$

8 a $\displaystyle \frac{f(z+h) - f(z)}{h} = \frac{Re(h)}{h} = \begin{cases} 1 & h \in \mathbb{R} \\ 0 & h = iy,\ y \in \mathbb{R} \end{cases}$

Thus, the limit does not exist.

b $\displaystyle \frac{f(z+h) - f(z)}{h} = \frac{Im(h)}{h} = \begin{cases} 0 & h \in \mathbb{R} \\ -i & h = iy,\ y \in \mathbb{R} \end{cases}$

Thus, the limit does not exist.

(9) If $z=0$, $\frac{\Delta w}{\Delta z} = \frac{\frac{\overline{\Delta z}^2}{\Delta z}}{\Delta z}$

$= 1$ if Δz is on real axis or imaginary axis.

If $\Delta y = \Delta x$, then $\frac{\Delta w}{\Delta z} = \left(\frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\right)^2$

$= \left(\frac{2}{-2i}\right)^2 = -1$

The limit approaching 0 on real axis and the line $x=y$ are different, thus the limit does not exist.

P103 (1a) $(1+i)^i = e^{i\log(1+i)}$

$= e^{i(\log\sqrt{2} + \frac{i\pi}{4} + 2\pi n i)}$

$= e^{-\pi/4 - 2\pi n} e^{i\frac{1}{2}\log 2}$ $n \in \mathbb{Z}$.

(1b) $\frac{1}{i^{2i}} = i^{-2i} = e^{-2i\log i}$

$= e^{-2i(\pi/2 i + 2n\pi i)}$ $n \in \mathbb{Z}$.

$= e^{(4n+1)\pi}$

(2a) $(-i)^i = e^{i\,\mathrm{Log}(-i)}$

$= e^{i(-\pi/2 i)}$

$= e^{\pi/2}$

2b $$\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i} = e^{3\pi i \,\mathrm{Log}\left(\frac{e}{2}(-1-\sqrt{3}i)\right)}$$

$$= e^{3\pi i\left(1-\frac{2\pi}{3}i\right)}$$

$$= -e^{2\pi^2}$$

2e $$(1-i)^{4i} = e^{4i\,\mathrm{Log}(1-i)}$$

$$= e^{4i\left(\log\sqrt{2} - \frac{\pi}{4}i\right)}$$

$$= e^{\pi}\left(e^{2i\log 2}\right).$$

3 $$(-1+\sqrt{3}i)^{3/2} = e^{3/2\log(-1+\sqrt{3}i)}$$

$$= e^{3/2\left(\log 2 + \frac{2\pi}{3}i + 2\pi n i\right)} \qquad n \in \mathbb{Z}$$

$$= 2\sqrt{2}\, e^{\pi i + 3\pi n i}$$

$$= \pm 2\sqrt{2}.$$

6 $$z^a = e^{a\mathrm{Log} z} = e^{a(\log|z| + i\,\mathrm{Arg}(z))}$$

$$|z^a| = e^{a\log|z|}$$

P109, 16 $\cos z = 2$

$$\cos x\cosh y - i\sin x\sinh y = 2$$

We have $\begin{cases} \cos x\cosh y = 2 \\ \sin x\sinh y = 0 \end{cases}$

5

Since $\sin x \sinh y = 0 \Rightarrow \begin{cases} x = \pi n \text{ or} \\ y = 0 \end{cases}$

If $x = \pi n$

$(-1)^n \cosh y = 2$, since $\cosh y$ is positive

n is even, then

$z = 2\pi n + i \cosh^{-1} 2$.

$$\cosh y = 2 = \frac{e^y + e^{-y}}{2}$$

$$(e^y)^2 - 4e^y + 1 = 0$$

$$e^y = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 + \sqrt{3}$$

$\Rightarrow z = 2n\pi + i \log(2 + \sqrt{3})$.